

# Lyapunov functions via Whitney's size functions

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## Abstract

In this paper we present a technique for constructing Lyapunov functions based on Whitney's size functions. Applications to asymptotically stable equilibrium points, isolated sets, expansive homeomorphisms and continuum-wise expansive homeomorphisms are given.

## 1 Introduction

In Dynamical Systems and Differential Equations it is important to determine the stability of trajectories and a well known technique for this purpose is to find a Lyapunov function. In order to fix ideas consider a continuous flow  $\phi: \mathbb{R} \times X \rightarrow X$  on a compact metric space  $(X, \text{dist})$  with a singular (or equilibrium) point  $p \in X$ , i.e.,  $\phi_t(p) = p$  for all  $t \in \mathbb{R}$ . A Lyapunov function for  $p$  is a continuous non-negative function that vanishes only at  $p$  and strictly decreases along the orbits close to  $p$ . Recall that  $p$  is *stable* if for all  $\varepsilon > 0$  there is  $\delta > 0$  such that if  $\text{dist}(x, p) < \delta$  then  $\text{dist}(\phi_t(x), p) < \varepsilon$  for all  $t \geq 0$ . We say that  $p$  is *asymptotically stable* if it is stable and there is  $\delta_0 > 0$  such that if  $\text{dist}(x, p) < \delta_0$  then  $\phi_t(x) \rightarrow p$  as  $t \rightarrow +\infty$ . The existence of a Lyapunov function for an equilibrium point implies the asymptotic stability of the equilibrium point.

A remarkable result, first proved by Massera in [6], is the converse: every asymptotically stable singular point admits a Lyapunov function. Later, other authors obtained Lyapunov functions with different methods, see for example [1, 2]. In [3] a generalization is proved in the context of arbitrary metric spaces. The purpose of the present paper is to develop a different technique that allows us to construct Lyapunov functions for different dynamical systems as: isolated sets, expansive homeomorphisms and continuum-wise expansive homeomorphisms. Our techniques are based on the size function  $\mu$  introduced by Whitney in [8].

In order to motivate our work let us show how to construct a Lyapunov function for an asymptotically stable singular point. Denote by  $\mathcal{K}(X)$  the set of non-empty compact subsets of  $X$ . In the set  $\mathcal{K}(X)$  we consider the Hausdorff distance  $\text{dist}_H$  making  $(\mathcal{K}(X), \text{dist}_H)$  a metric space. Recall that

$$\text{dist}_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ and } B \subset B_\varepsilon(A)\},$$

where  $B_\varepsilon(C) = \cup_{x \in C} B_\varepsilon(x)$  and  $B_\varepsilon(x)$  is the usual ball of radius  $\varepsilon$  centered at  $x$ . See [7] for more on the Hausdorff metric. A *size function* is a continuous map  $\mu: \mathcal{K}(X) \rightarrow \mathbb{R}$  satisfying:

1.  $\mu(A) \geq 0$  with equality if and only if  $A$  has only one point,

2. if  $A \subset B$  and  $A \neq B$  then  $\mu(A) < \mu(B)$ .

In [8] it is proved that size functions exists for every compact metric space.

**Theorem 1.1.** *If  $\phi$  is a continuous flow on  $X$  with an asymptotically stable singular point  $p$  then there are an open set  $U$  containing  $p$  and a continuous function  $V: U \rightarrow \mathbb{R}$  satisfying:*

1.  $V(x) \geq 0$  for all  $x \in U$  with equality if and only if  $x = p$  and
2. if  $t > 0$  and  $\{\phi_s(x) : s \in [0, t]\} \subset U$  then  $V(\phi_t(x)) < V(x)$ .

*Proof.* By the conditions on  $p$  there are  $\delta_0, \delta > 0$  such that if  $\text{dist}(x, p) < \delta$  then  $\phi_t(x) \in B_{\delta_0}(p)$  for all  $t \geq 0$  and  $\phi_t(x) \rightarrow p$  as  $t \rightarrow \infty$ . Define  $U = B_\delta(p)$  and  $V: U \rightarrow \mathbb{R}$  as

$$V(x) = \mu(\{\phi_t(x) : t \geq 0\} \cup \{p\})$$

where  $\mu$  is a size function. Since  $\phi_t(x) \rightarrow p$  we have that

$$O(x) = \{\phi_t(x) : t \geq 0\} \cup \{p\} \tag{1}$$

is a compact set for all  $x \in U$ . Notice that if  $t > 0$  then  $O(\phi_t(x)) \subset O(x)$  and the inclusion is proper. Therefore,  $V(\phi_t(x)) < V(x)$  because  $\mu$  is a size function. Also notice that  $V(p) = 0$  and  $V(x) > 0$  if  $x \neq p$ . In order to prove the continuity of  $V$ , we will prove the continuity of  $O: U \rightarrow K(X)$ , the map defined by (1). Since  $\mu$  is continuous we will conclude the continuity of  $V$ .

Let us prove the continuity of  $O$  at  $x \in U$ . Take  $\varepsilon > 0$ . By the asymptotic stability of  $p$  there are  $\rho, T > 0$  such that if  $y \in B_\rho(x)$  then  $\phi_t(y) \in B_{\varepsilon/2}(p)$  for all  $t \geq T$ . By the continuity of the flow, there is  $r > 0$  such that if  $y \in B_r(x)$  then  $\text{dist}(\phi_t(x), \phi_t(y)) < \varepsilon$  for all  $t \in [0, T]$ . Now it is easy to see that if  $y \in B_{\min\{\rho, r\}}(x)$  then  $\text{dist}_H(O(x), O(y)) < \varepsilon$ , proving the continuity of  $O$  at  $x$  and consequently the continuity of  $V$ .  $\square$

Let us recall that size functions can be easily defined. A variation of the construction given in [8], adapted for compact metric spaces, is the following. Let  $q_1, q_2, q_3, \dots$  be a sequence dense in  $X$ . Define  $\mu_i: \mathcal{K}(X) \rightarrow \mathbb{R}$  as

$$\mu_i(A) = \max_{x \in A} \text{dist}(q_i, x) - \min_{x \in A} \text{dist}(q_i, x).$$

The following formula defines a size function  $\mu: \mathcal{K}(X) \rightarrow \mathbb{R}$

$$\mu(A) = \sum_{i=1}^{\infty} \frac{\mu_i(A)}{2^i},$$

as proved in [8]. In Section 2 we extend Theorem 1.1 by constructing a Lyapunov function for an isolated invariant sets.

For the study of expansive homeomorphisms (see Definition 3.1) Lewowicz introduced in [5] Lyapunov functions. He proved that expansiveness is equivalent with the existence of such function. In Section 3 we give a different proof of this result by constructing a Lyapunov function defined for compact subsets of the space. In [4] Kato introduced another form of expansiveness called continuum-wise expansiveness (see Definition 3.2). With our techniques we prove that continuum-wise expansiveness is equivalent with the existence of a Lyapunov function on continua subsets of the space.

## 2 Lyapunov Functions for Isolated Sets

In this section we consider continuous flows on compact metric spaces. The purpose is to construct a Lyapunov for an isolated set of the flow using a size function. First we consider the case of an isolated set consisting of a point.

### 2.1 Isolated Singularities

Let  $\phi$  be a continuous flow on a compact metric space  $(X, \text{dist})$ . A point  $p \in X$  is *singular* for  $\phi$  if  $\phi_t(p) = p$  for all  $t \in \mathbb{R}$ . A singular point  $p \in X$  is *isolated* if there is an open *isolating neighborhood*  $U$  of  $p$  such that if  $\phi_{\mathbb{R}}(x) \subset U$  then  $x = p$ .

**Definition 2.1.** An open set  $U$  is an *adapted neighborhood* of an isolated singular point  $p \in U$  if for every orbit segment  $l \subset \text{clos}(U)$  with extreme points in  $U$  it holds that  $l \subset U$ .

Given a set  $A \subset X$  and  $x \in A$  denote by  $\text{comp}_x(A)$  the connected component of  $A$  that contains the point  $x$ .

**Proposition 2.1.** *Every isolated singular point has an adapted neighborhood.*

*Proof.* Let  $r > 0$  be such that  $\text{clos}(B_r(p))$  is contained in an isolating neighborhood of  $p$ . For  $\rho \in (0, r)$  define the set

$$U_\rho = \{x \in B_r(p) : \text{comp}_x(\phi_{\mathbb{R}}(x) \cap B_r(p)) \cap B_\rho(p) \neq \emptyset\}.$$

By the continuity of the flow we have that  $U_\rho$  is an open set for all  $\rho \in (0, r)$ . Let us prove that if  $\rho$  is sufficiently small then  $U_\rho$  is an adapted neighborhood. By contradiction, suppose that there are  $\rho_n \rightarrow 0$ ,  $a_n, b_n \in U_{\rho_n}$ ,  $t_n \geq 0$  such that  $b_n = \phi_{t_n}(a_n)$  and  $l_n = \phi_{[0, t_n]}(a_n) \subset \text{clos}(U_{\rho_n})$  but  $l_n$  is not contained in  $U_{\rho_n}$ . Then there is  $s_n \in (0, t_n)$  such that  $\phi_{s_n}(a_n) \in \partial B_r(p)$ . Also, there must be  $u_n < 0$  and  $v_n > 0$  such that  $\phi_{u_n}(a_n), \phi_{v_n}(b_n) \in B_{\rho_n}(p)$ . But a limit point of  $\phi_{s_n}(a_n)$  contradicts that  $\text{clos}(B_r(p))$  is contained in an isolating neighborhood of  $p$ .  $\square$

Fix an isolated point  $p$  with an adapted neighborhood  $U$ . Consider the sets

$$\begin{aligned} W_U^s(p) &= \{x \in U : \lim_{t \rightarrow +\infty} \phi_t(x) = p \text{ and } \phi_{\mathbb{R}^+}(x) \subset U\}, \\ W_U^u(p) &= \{x \in U : \lim_{t \rightarrow -\infty} \phi_t(x) = p \text{ and } \phi_{\mathbb{R}^-}(x) \subset U\}, \end{aligned}$$

For  $x \in U$  define the orbit segments

$$\begin{aligned} O_U^+(x) &= \text{comp}_x(U \cap \phi_{[0, +\infty)}(x)), \\ O_U^-(x) &= \text{comp}_x(U \cap \phi_{(-\infty, 0]}(x)). \end{aligned}$$

Define  $C = X \setminus U$  and let  $V_p^+, V_p^- : U \rightarrow \mathcal{K}(X)$  be defined as

$$\begin{cases} V_p^+(x) = \text{clos}(O_U^+(x) \cup W_U^u(p)) \cup C, \\ V_p^-(x) = \text{clos}(O_U^-(x) \cup W_U^s(p)) \cup C. \end{cases}$$

**Definition 2.2.** A *Lyapunov function* for an isolated point  $p$  is a continuous map  $V: U \rightarrow \mathbb{R}$  defined in a neighborhood of  $p$  such that if  $t > 0$  and  $\phi_{[0,t]}(x) \subset U \setminus \{p\}$  then  $V(x) > V(\phi_t(x))$ .

**Theorem 2.2.** If  $p$  is an isolated point and  $U$  is an adapted neighborhood of  $p$  then the maps  $V_p^+$  and  $V_p^-$  are continuous in  $U$ . If in addition,  $\mu$  is a size function on  $\mathcal{K}(X)$  then  $V: U \rightarrow \mathbb{R}$  defined as

$$V(x) = \mu(V_p^+(x)) - \mu(V_p^-(x))$$

is a *Lyapunov function* for  $p$ .

*Proof.* Let us prove the continuity of  $V_p^+$  by contradiction. Assume that  $x_n \rightarrow x \in U$  and  $V_p^+(x_n) \rightarrow K$  with the Hausdorff distance but  $K \neq V_p^+(x)$ . By definitions we have that

$$\text{clos}(W_U^u(p)) \cup C \subset K \cap V_p^+(x). \quad (2)$$

Recall that  $C$  was defined as the complement of  $U$  in  $X$ . Take a point  $y \in K \setminus V_p^+(x) \cup V_p^+(x) \setminus K$ . By the inclusion (2) we know that  $y \notin \text{clos}(W_U^u(p)) \cup C$ . We divide the proof in two cases.

*Case 1.* Suppose first that  $y \in K \setminus V_U^+(x)$ . Since  $y \in K$  there is a sequence  $t_n \geq 0$  such that  $\phi_{t_n}(x_n) \rightarrow y$  and  $\phi_{[0,t_n]}(x_n) \subset U$ . If  $t_n \rightarrow \infty$  then  $x \in W_U^s(p)$ . Consequently,  $y \in W_U^u(p)$ , which is a contradiction. Therefore  $t_n$  is bounded. Without loss of generality assume that  $t_n \rightarrow \tau \geq 0$  and then  $\phi_\tau(x) = y$ . Thus  $\phi_{[0,\tau]}(x) \subset \text{clos}(U)$ . Since  $y \notin C$  we have that  $y \in U$ . Now, since  $U$  is an adapted neighborhood we conclude that  $\phi_{[0,\tau]}(x) \subset U$  and then  $y \in O^+(x) \subset V_p^+(x)$ . This contradiction finishes this case.

*Case 2.* Now assume that  $y \in V_p^+(x) \setminus K$ . In this case we have that  $y = \phi_s(x)$  for some  $s \geq 0$  and  $\phi_{[0,s]}(x) \subset U$ . Then  $\phi_s(x_n) \rightarrow y$  and  $y \in K$ . This contradiction proves that  $V_p^+$  is continuous in  $U$ .

The continuity of  $V_p^-$  is proved in a similar way. Let us show that  $V$  is a Lyapunov function for  $p$ . The continuity of  $V$  in  $U$  follows by the continuity of  $V_p^+$ ,  $V_p^-$  and the size function  $\mu$ .

Now take  $x \notin U \setminus \{p\}$ . We will show that  $V$  decreases along the orbit segment of  $x$  contained in  $U$ . Notice that for all  $t > 0$ ,  $O_U^+(\phi_t(x)) \subset O_U^+(x)$  if  $\phi_{[0,t]}(x) \subset U$ . Therefore  $V_p^+(\phi_t(x)) \leq V_p^+(O^+(x))$ . The equality can only hold if  $x \in W_U^u(p)$ . But in this case we have that  $x \notin W_U^s(p)$  because  $W_U^u(p) \cap W_U^s(p) = \{p\}$ . Then  $V_p^-(\phi_t(x)) > V_p^-(x)$ . Therefore,  $V(\phi_t(x)) < V(x)$  and  $V$  is a Lyapunov function for  $p$ .  $\square$

## 2.2 Isolated Sets

Let  $\phi: \mathbb{R} \times X \rightarrow X$  be a continuous flow on a compact metric space  $X$ . Consider a  $\phi$ -invariant set  $\Lambda \subset X$ , i.e.,  $\phi_t(\Lambda) = \Lambda$  for all  $t \in \mathbb{R}$ . We say that  $\Lambda$  is an *isolated set* with *isolating neighborhood*  $U$  if  $\phi_{\mathbb{R}}(x) \subset U$  implies  $x \in \Lambda$ .

**Definition 2.3.** A *Lyapunov function* for an isolated set  $\Lambda$  is a continuous function  $V: U \rightarrow \mathbb{R}$  defined on an open set  $U$  containing  $\Lambda$  such that:

1.  $V(x) = 0$  if and only if  $x \in \Lambda$ ,
2. if  $\phi_{[0,t]}(x) \subset U \setminus \Lambda$  then  $V(x) > V(\phi_t(x))$ .

Let us show how the construction of a Lyapunov function for an isolated set can be reduced to the case of an isolated singular point.

**Theorem 2.3.** *Every isolated set admits a Lyapunov function.*

*Proof.* Consider the set  $Y = (X \setminus \Lambda) \cup \{\Lambda\}$ . On  $Y$  define the distance  $d$  as

$$d(x, y) = \min\{\text{dist}(x, y), \text{dist}(x, \Lambda) + \text{dist}(y, \Lambda)\}.$$

It is easy to see that  $(Y, d)$  is a compact metric space. Also, the flow  $\phi$  induces naturally a flow  $\phi'$  on  $Y$  with  $\Lambda$  as an isolated singular point. Consider from Theorem 2.2 a Lyapunov function for  $\Lambda$  as an isolated singular point of  $\phi'$ . This function naturally defines a Lyapunov function for  $\Lambda$  as an isolated set of  $\phi$ .  $\square$

### 3 Applications to homeomorphisms

Let  $f: X \rightarrow X$  be a homeomorphism of a compact metric space  $(X, \text{dist})$ . An  $f$  invariant set  $\Lambda$  is *isolated* if there is an open neighborhood  $U$  of  $\Lambda$  such that  $f^n(x) \in U$  for all  $n \in \mathbb{Z}$  implies that  $x \in \Lambda$ .

**Theorem 3.1.** *Every isolated set  $\Lambda$  for a homeomorphism  $f$  admits a Lyapunov function, that is, a continuous map  $V: U \subset X \rightarrow \mathbb{R}$  defined on a neighborhood of  $\Lambda$  such that:*

1.  $V(x) = 0$  if and only if  $x \in \Lambda$ ,
2.  $V(x) > V(f(x))$  if  $x, f(x) \in U \setminus \Lambda$ .

*Proof.* Consider  $\phi: \mathbb{R} \times X_f \rightarrow X_f$  the suspension of  $f$ . Consider  $i: X \rightarrow X_f$  a homeomorphism onto its image such that  $i(X)$  is a global cross section of  $\phi$ . It is easy to see that  $\Lambda$  is an isolated set for  $f$  if and only if  $\Lambda_f = \phi_{\mathbb{R}}(i(\Lambda))$  is an isolated set for  $\phi$ . Now consider a Lyapunov function  $V'$  for  $\Lambda_f$ . A Lyapunov function for  $f$  can be defined by  $V(x) = V'(i(x))$ .  $\square$

**Definition 3.1.** A homeomorphism  $f: X \rightarrow X$  of a compact metric space is *expansive* if there is  $\alpha > 0$  (an *expansive constant*) such that if  $x \neq y$  then there is  $n \in \mathbb{Z}$  such that  $\text{dist}(f^n(x), f^n(y)) > \alpha$ .

Recall that  $\mathcal{K}(X)$  denotes the compact metric space of compact subsets of  $X$  with the Hausdorff metric. Denote by  $\mathcal{F}_1 = \{A \in \mathcal{K}(X) : |A| = 1\}$  where  $|A|$  denotes the cardinality of  $A$ . Given a homeomorphism  $f: X \rightarrow X$  define the homeomorphism  $f': \mathcal{K}(X) \rightarrow \mathcal{K}(X)$  as  $f'(A) = \{f(x) : x \in A\}$ . Notice that  $\mathcal{F}_1$  is invariant under  $f'$ .

**Corollary 3.2.** *For a homeomorphism  $f: X \rightarrow X$  the following statements are equivalent:*

1.  $f$  is an expansive homeomorphism,
2.  $\mathcal{F}_1$  is an isolated set for  $f'$ ,
3. there is a continuous function  $V: U \subset \mathcal{K}(X) \rightarrow \mathbb{R}$  defined on a neighborhood of  $\mathcal{F}_1$  such that  $V(A) = 0$  if and only if  $A \in \mathcal{F}_1$  and  $V(A) > V(f'(A))$  if  $A, f'(A) \in U \setminus \mathcal{F}_1$ .

*Proof.* (1  $\rightarrow$  2). Let  $\delta$  be an expansive constant and define

$$U = \{A \in \mathcal{K}(X) : \text{diam}(A) < \delta\}.$$

It is easy to see that  $U$  is an isolating neighborhood of  $\mathcal{F}_1$ .

(2  $\rightarrow$  3). It follows by Theorem 3.1.

(3  $\rightarrow$  1). Take  $\delta > 0$  such that if  $\text{dist}(x, y) \leq \delta$  then  $\{x, y\} \in U$ . Let us prove that  $\delta$  is an expansive constant for  $f$ . Assume by contradiction that  $\text{dist}(f^n(x), f^n(y)) \leq \delta$  for all  $n \in \mathbb{Z}$  and  $x \neq y$ . Define  $A = \{x, y\}$ . We have that  $V(f^n(A))$  is a decreasing sequence. Without loss of generality assume that  $V(A) < 0$ . Suppose that  $f^n(A)$  accumulates in  $B$ . Now it is easy to see that  $B \in U \setminus \mathcal{F}_1$  and also  $V(B) = V(f'(B))$ . This contradiction proves the theorem.  $\square$

Recall that a *continuum* is a compact connected set. Denote by  $\mathcal{C}(X) = \{C \in \mathcal{K}(X) : C \text{ is connected}\}$  the space of continua of  $X$ .

**Definition 3.2.** A homeomorphism  $f: X \rightarrow X$  is *continuum-wise expansive* if there is  $\delta > 0$  such that if  $C \in \mathcal{C}(X)$  and  $\text{diam}(f^n(C)) \leq \delta$  for all  $n \in \mathbb{Z}$  then  $C \in \mathcal{F}_1$ .

A *Lyapunov function* for a continuum-wise expansive homeomorphism is a continuous function  $V: U \subset \mathcal{C}(X) \rightarrow \mathbb{R}$  defined on a neighborhood of  $\mathcal{F}_1(X)$  in  $\mathcal{C}(X)$  such that  $V(\{x\}) = 0$  for all  $x \in X$  and  $V(f(C)) < V(C)$  if  $C \notin \mathcal{F}_1$  and  $C, f(C) \in U$ .

**Corollary 3.3.** For a homeomorphism  $f: X \rightarrow X$  the following statements are equivalent:

1.  $f$  is a continuum-wise expansive homeomorphism,
2.  $\mathcal{F}_1$  is an isolated set for  $f': \mathcal{C}(X) \rightarrow \mathcal{C}(X)$ ,
3. there is a continuous function  $V: U \subset \mathcal{C}(X) \rightarrow \mathbb{R}$  defined on an open set  $U \subset \mathcal{C}(X)$  containing  $\mathcal{F}_1$  such that  $V(A) = 0$  if and only if  $A \in \mathcal{F}_1$  and  $V(A) > V(f'(A))$  if  $A, f'(A) \in U \setminus \mathcal{F}_1$ .

*Proof.* The proof is similar to the proof of Corollary 3.2.  $\square$

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